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Proportional relationships decluttered—at last!

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Introduction

The beauty of the mathematical idea of proportionality is too easily lost in a clutter of procedures and rules. As such, it may seem mysterious to students despite the best efforts of teachers. Yet, as Lesh, Post, & Behr (1988) contend, proportional reasoning is the capstone of arithmetic and the cornerstone of advanced mathematics (p. 94). Given this importance, it is essential to examine how we might build strong ways of thinking to help students better understand proportional relationships.

In this article we attempt to declutter proportionality. We do not approach thinking about proportional relationships as simply setting up and solving proportional equations! Instead, we want students to think about varying quantities and to investigate patterns so they can describe quantities in terms of proportional relationships. Students encounter proportional relationships in their everyday world, and we provide examples of where these relationships surface and how we can approach building strong ways of thinking. We also recognize that this is just the beginning of a longer journey towards developing a robust understanding of proportionality.

Additionally, the student experiences we describe highlight the power of modeling with mathematics. After all, the story of proportional relationships begins with the practice of collating data from real-world scenarios, recognizing a particular structure in data that is common to many scenarios, and then using that insight about structure to solve problems. Then, once students' algebraic skills have evolved, the kicker emerges: one comes to realize that this identified structure can be perfectly encapsulated and described by a single equation of the form $y = rx$. The stunning power of modeling with mathematics is now brought to the forefront!

This simple and profound story, however, is typically obscured. Our work here is to declutter this beautiful and important subject.

The setup

Many common scenarios in the world, in everyday life, and in mathematics involve two or more quantities that

i) we can naturally measure,

(a count of objects, a length, a temperature, or a time period, for example)

ii) whose measures can or do adopt a variety of possible values,

(one can imagine buying different counts of apples or one can run an experiment for different lengths of time, for instance)

and

iii) whose measures seem related to each other.

(The number of dollars spent depends on the number of apples bought: as I buy more apples, I spend more dollars. The area of a circle depends on the length of its radius: if I decrease the radius, I will also decrease the area.)

Moreover, it is not uncommon for us to recognize scenarios involving two measurable quantities whose measures vary in a particularly straightforward way:

iv) the measures "scale in tandem."

By this we mean that if we double the measure of one quantity, the measure of the second doubles as well; if we triple the measure of one quantity, the measure of the second triples as well; if we halve the measure of one quantity, the measure of the other halves as well; and so on.

Common sense usually allows us to determine when this is or is not the case for a given scenario.

Scenario: Consider a stack of identical books. The height of the stack (as measured in cm, say) and the count of books (measured as a number) are two quantities we can measure, whose measures can take on a variety of different values, with values that are related to each, and related specifically via scaling in tandem: to double the height of a stack, double the count of books; changing the count of books by a factor of ten gives a new stack ten times the height; and so on.

Scenario: At present the euros to US dollar exchange rate is €100 = $114. The count of euros one might have and the matching count of US dollars (USD) are quantities we can measure, measures that can take on varying values, with values that are related to each other, and related specifically via scaling in tandem: €200 in my pocket (double the stated figure) corresponds to $228; and $19 (one-sixth of 114) is worth €16.67 (one-sixth of 100, up to rounding).

Scenario: I walk away from home, directly east, at a constant speed of 5 miles per hour, without a break. After one hour of walking my distance from home will be 5 miles. After 1.5 hours of walking, my distance from home will be 1.5 times as much as after one hour, namely, 7.5 miles. After 24 hours of walking, I'll be 24 times as much as after one hour, or 120 miles from home.

Scenario: If a count of 720 people represents 60% of a town's population, then half of that, 360 people, represents 30%, and a third of this, namely, 120 people, represents 10% of the town's population. I now readily deduce that ten times this count, namely, 1200 people, is the entire town's population.

In each of these four scenarios we can indeed identify two quantities we can measure, whose measures can take on varying values, with values that are related to each other, and related specifically via scaling in tandem.

Scenario: Central Park is 2.5 miles long.

This is a fine fact, but it is a statement only about the measure of one specific quantity that cannot adopt different values.

Scenario: Anu is 5.5 feet tall and Benu is 6.1 feet tall.

The measures of two quantities are mentioned here, but there is no indication in this scenario that these measures can or should be considered to change.

Scenario: The ratio of boys to girls in a class is 3:5. There are 16 students in the class.

Two measures are implied here—a count of boys and a count of girls—but the statement that the total count of students is fixed means that the measures of these two counts is static: there are 6 boys and 10 girls. The number of students does not seem to be changing.

However, if we were simply presented the scenario, "In a certain school, each class has a 3:5 ratio of boys to girls," then the count of boys and count of girls are measures that can adopt different values: one class could have 3 boys and 5 girls; another class could have 9 boys and 15 girls; another class 30 boys and 50 girls; and so forth. (And later in this article we will demonstrate that the data here—the count of boys in a class and the matching count of girls—does indeed scale in tandem.)

Scenario: Consider the areas and the perimeters of all the squares one can draw.

Here we have two varying, measurable quantities—an area and a length—which are related to each other, but they do not scale in tandem: if we double the perimeter of a square, then its side length doubles to give a new square of quadruple the area, for instance.

P = 4   P = 8
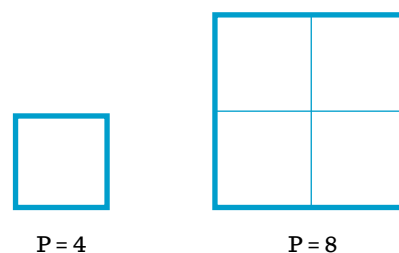
Scenario: Consider a stack of identical books sitting on a table and the distance from the top of the stack to the floor. Again, we have two varying, measurable quantities—a count of books and the distance from the top of the stack to the floor—that are related to each other, but not via scaling in tandem, alas. If I double the count of books, the distance of the top of the stack to the floor does not double.

Scenario: If it takes 6 cats 4 hours to catch 7 rats, then presumably it would take 12 cats half the time to catch 7 rats (2 hours), but it would take 3 cats double the time (8 hours) to catch 7 rats. The count of cats and the number of hours needed to catch 7 rats are not scaling in tandem. (When one doubles, the other halves, for instance.)

Scenario: If it takes 22 minutes for 1 sock to dry when hung on a clothesline, then 2 socks will still take 22 minutes to dry!

Scenario: No matter how many carrots I eat in a day, my shoe size does not change. (There is no reason that the count of carrots I eat in any one day should affect my shoe size for that day.)

Definition[1]: Two measurable quantities in a given scenario that can, or do, vary in value are said to be in a *proportional relationship* if they "scale in tandem," that is, if the measure of one quantity in the scenario changes by a factor k, then the measure of the other quantity is sure to also change by the same factor k.

And just to be clear, people might use the term *proportional reasoning* to mean the act of identifying two measurable quantities in a scenario whose measures covary via a proportional relationship, along with the subsequent act(s) of using mathematical techniques and one's common sense to answer questions about that scenario.

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[1] See Thompson & Saldanha, 2003.

A worked example

When presented with a scenario of some kind, ask:

1. Are there two measurable quantities in the scenario of interest that can, or do, adopt a variety of different values?

If so, next ask:

2. Does it seem reasonable to presume these measures scale in tandem?

If so, then you are all set to use common sense to answer sophisticated questions about that scenario.

Example

I can buy 27 Kewpie dolls for $15.

a) How many dolls can I buy for $1,000? How many can I buy for $1,200? How many can I buy for $2,335 and for $330 and for $7,000 and for $850 and for a million dollars?

b) Sameer bought 549 dolls. How much did he spend?

Answer

We certainly have two measurable quantities in this situation: a count of Kewpie dolls purchased and a dollar value of that expense.

One might argue that in this scenario these two quantities are probably NOT in a proportional relationship: one usually gets a discount with bulk purchases, so ordering a thousand-fold dolls is likely not to cost you 1,000 times as much. (You might get a 20% discount, perhaps.)

But let's put that caveat aside and assume that purchasing three times as many dolls costs three times as much, purchasing 100 times as many dolls, 100 times as much, and so on. That is, let's assume we do have a proportional relationship.

Relationships between two quantities are often presented in tables or with equal signs, but to help avoid mixing new ideas with preconceived

notions we introduce, temporarily, a new symbol: the double arrow. You might prefer to use an equal sign instead of a double arrow or present your data in a table or with a pair of number lines. Whatever is sensible and makes sense to you is fine!

Here is the one data pair we are given.

$$27 \text{ dolls} \longleftrightarrow 15 \text{ dollars}$$

Using scaling in tandem, we also have

$$54 \text{ dolls} \longleftrightarrow 30 \text{ dollars}$$
$$540 \text{ dolls} \longleftrightarrow 300 \text{ dollars}$$
$$1,620 \text{ dolls} \longleftrightarrow 900 \text{ dollars}$$

and so on.

But we want the count of dolls that "goes with" $1,000. So let's scale in tandem with intention.

We have

$$27 \text{ dolls} \longleftrightarrow 15 \text{ dollars}$$

To convert $15 dollars into $1,000 we can scale by one-third and then scale by 200.

$$9 \text{ dolls} \longleftrightarrow 5 \text{ dollars}$$
$$1,800 \text{ dolls} \longleftrightarrow 1,000 \text{ dollars}$$

So we can purchase 1,800 dolls for $1,000.

But the question wants us to keep going for different dollar amounts. This work is going to get awfully tedious. Can I come up with a general formula for how many dolls I can purchase for *k* dollars?

Well, we had

$$9 \text{ dolls} \longleftrightarrow 5 \text{ dollars}$$

Scaling this by one-fifth gives

$$1.8 \text{ dolls} \longleftrightarrow 1 \text{ dollar}$$

Reality is a little weird here. They will certainly not sell me fractional dolls. But let's carry on and assume the manufacturer will handle this in some nice way for us. (Maybe the rule is that Kewpie dolls must be bought in sets of nine?)

Scaling by k we have

$$1.8k \text{ dolls} \longleftrightarrow k \text{ dollars}$$

There we have it, a general formula. We can purchase 1.8k dolls for \$$k$. Now we can complete part a) of the question if we wish.

Actually, we can complete part b), too! We just need to solve 549 = 1.8k.

$$k = \frac{549}{1.8} = \frac{9 \times 61}{9 \times .02} = 61 \times 5 = 305$$

Doing so shows that Sameer spent \$305 to buy 549 dolls.

A few common places where proportional relationships arise

Here we present various scenarios where common sense informs us that a proportional relationship is at play.

1. Unit conversion (exchange rates)

A length of 100 inches matches a length of 254 cm.

$$100 \text{ inches} \longleftrightarrow 254 \text{ cm}$$

If we scale the count of inches by a factor k, then common sense tells us the count of centimeters measured will scale by k as well.

To find the measure of a one-meter (100 cm) length, say, in inches, we can scale by a factor of $\frac{1}{254}$ and then by a factor of 100.

$$\frac{100}{254} \text{ inches} \longleftrightarrow 1 \text{ cm}$$

$$\frac{10000}{254} \text{ inches} \approx 39.4 \text{ in} \longleftrightarrow 100 \text{ cm} = 1\text{m}$$

At present, 100 euros have a value 114 US dollars.

$$€100 \longleftrightarrow \$114$$

To find the value of €365, for example, we can use $\frac{365}{100}$ as a scale factor.[2]

$$€365 \longleftrightarrow \frac{365}{100} \times 144 = \$146.10$$

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[2] Or scale by $\frac{1}{100}$ then scale by 365.

2. Constant rates

A scenario that describes a constant rate of some kind usually leads to a proportional relationship.

Suppose that I am taxed at a constant rate of 35 cents for every dollar earned.

$$\$1 \text{ earned} \longleftrightarrow \$0.35 \text{ for the IRS}$$

If I earned \$44,230 last year, I will pay \$44,230 × 0.35 = \$15,480.50 in taxes.

$$\$44230 \text{ earned} \longleftrightarrow \$15480 \text{ for the IRS}$$

The road to Wagga Wagga rises at a constant rate of 100 feet for every mile driven along the road.

$$1 \text{ mile} \longleftrightarrow 100 \text{ feet}$$

After how many miles of driving will I have increased in elevation by one mile?

We can scale by a factor of 52.8.

$$52.8 \text{ miles} \longleftrightarrow 5280 \text{ feet}$$

Answer: 52.8 miles.

A train moves at a constant speed of 65 mph.

$$1 \text{ hour} \longleftrightarrow 65 \text{ miles traversed}$$

How long will it take to traverse 200 miles?

$$\frac{200}{65} \approx 3.08 \text{ hours} \longleftrightarrow 200 \text{ miles}$$

Answer: About three hours and five minutes.

Comment: There is a subtle point to be made here. The data being collected in this scenario of a train traveling at a constant rate is the time *spent traveling* and the distance *traversed* during that time and not an actual time of day and an actual distance from a fixed location.

For example, if I board the train at the station located 5 miles from home at 3:00 p.m., my distance from home is not in a proportional relationship with the hour shown on my clock. In general, a statement of a constant rate of change is usually a statement about changes in quantities. As an absurd example, "a constant increase of 2 floogles per neeb" informs us that

$$\begin{array}{c} \text{A change of} \\ \text{2 floogles} \end{array} \longleftrightarrow \begin{array}{c} \text{A change of} \\ \text{1 neeb} \end{array}$$

and that these changes scale in tandem. (Thus a change of 4 floogles matches a change of 2 neebs, and so on.) We do not know anything about the actual count of floogles and actual count of neebs at any instant.

Question: I will tell you some specific values for the quantities of floogles and neebs. Last week I had 17 floogles and 12 neebs. I watered them and this week I counted 34 floogles in my closet. How many neebs should I expect to count?

3. Percentages

135% of a quantity is 156. What is the quantity?

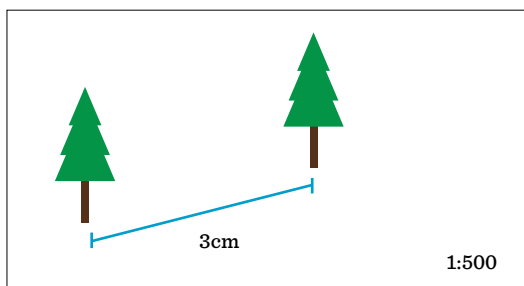$$135\% \longleftrightarrow \text{value } 156$$

$$1\% \longleftrightarrow \text{value } \frac{156}{135}$$

$$100\% \longleftrightarrow \text{value } \frac{15600}{135} = 115\frac{5}{9}$$

4. Scaled drawings

A scaled drawing or a map provides a scenario with two measurable quantities: the distances on the drawing and the distances in the real world. Each drawing comes with an indication of "scale." For instance, seeing 1:500 printed in the corner of a map indicates that one unit of measure on the map matches 500 units of measure in the real world.

$$1 \text{ unit on map} \longleftrightarrow 500 \text{ miles in real world}$$

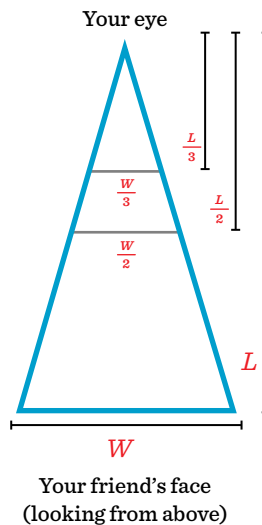If the distance between two particular trees on the map is 3 cm, how far apart are those trees in real life?

$$3 \text{ cm on map} \longleftrightarrow 1500 \text{ cm in real world}$$

Answer: The trees are 15 meters apart.

5. Similar figures in geometry

Stand in front of a friend, face-to-face, just a couple of feet apart. Raise your hands at about the halfway point and use the index finger of each hand to judge the width of your friend's face: line your left index finger with the left edge of your friend's face, your right index finger with the right edge of her face. (You might need to close one eye to do this.) Then look at the distance between your two fingers: that distance is half the width of your friend's face!
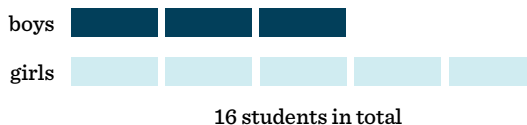
Similar triangles (and similar figures, in general) in geometry yield relationships between lengths that scale in tandem.

Your friend's face
(looking from above)

6. Ratios

Loosely speaking, two quantities in a scenario are said to come in an "a to b ratio," written $a{:}b$, if whenever a groups of the first quantity appear in the scenario, then b groups of the second quantity also appear.

For example, let's return to our boy-to-girl student example. If a class of 16 students has a boy-to-girl ratio of 3:5, then we know if we divide the boys into three same-sized groups, there are five matching same-sized groups of girls.

16 students in total

In this example there are eight groups in total, so we deduce that the group size is two and that there are thus six boys and ten girls.

Ratios often appear in "static scenarios" where no quantities are presumed to vary in measure, simply that the measures of one or two quantities are unknown and are to be deduced.

But a scenario could describe two quantities whose measures vary in "constant ratio." This does lead to a proportional relationship.

Example

In a certain school all classes have a boy-to-girl ratio of 3:5.

This scenario does lead to data: the count of boys and the matching count of girls in all possible classes. By imagining different group sizes in this diagram, we get the following data:

Group size 1	3 boys	$\longleftrightarrow$	5 girls
Group size 2	6 boys	$\longleftrightarrow$	10 girls
Group size 3	9 boys	$\longleftrightarrow$	15 girls
Group size 4	12 boys	$\longleftrightarrow$	20 girls

In general, we have:

$$\text{Group size } N: \quad 3N \text{ boys} \longleftrightarrow 5N \text{ girls}$$

Does this data scale in tandem?

Suppose we change the number of boys in a class by a factor k, from $3N$ boys to $3Nk$ boys. Must the count of girls change by that factor, too, to $5Nk$ girls?

Well, $3Nk$ boys corresponds to three groups of size Nk. (We have $3Nk = 3(Nk)$.) So there must be five groups of girls of this size, indeed making for $5(Nk) = 5Nk$ girls.

$$3N \text{ boys} \longleftrightarrow 5N \text{ girls}$$

$$(3N) \times k \text{ boys} \longleftrightarrow (5N) \times k \text{ girls}$$

As a result, k represents the scale factor of boys to girls. The data is scaling in tandem and we have a proportional relationship between the count of boys and the count of girls.

Tables and double number lines

The double-arrow notation can be handy, but others might prefer to present their data in a table.

Here's a table of the measures of two quantities, *A* and *B*.

A	B
5	7
10	14
30	42
3	4.2
6	8.4

Using the double-arrow notation, the first line can be read as

$$\text{5 units of quantity } A \longleftrightarrow \text{7 units of quantity } B$$

and we see that each line of this table does appear as a co-scaled version of this first relation.

Some people like to turn the table sideways and imagine the table of values along two parallel number lines.
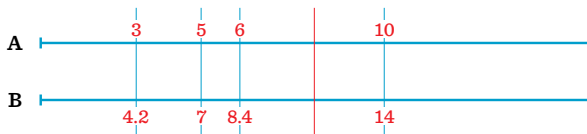
Here are the two number lines with the first data pair aligned on them.

This suggests that, on the "*A* number line," five units of distance from the left matches seven units of distance from the left on the "*B* number line." If we double the data values, these should give matching positions on the number lines of double the distances.

And if we continue scaling in this way—tripling the data values and tripling the distances along each number line; halving the data values and halving the distances along each number line; and so on—then we should have two parallel number lines whose values along it match all possible data values one might write in the table.

This allows us to eyeball additional matching data values. For example, we might argue that 8 on the *A* number line matches 11.2 on the *B* number line.

Check: Use scaling in tandem from

$$
\begin{array}{ccc}
5 \text{ units of} & \longleftrightarrow & 7 \text{ units of} \\
\text{quantity } A & & \text{quantity } B
\end{array}
$$

to get

$$
\begin{array}{ccc}
\frac{8}{5} \times 5 \text{ units of} & \longleftrightarrow & \frac{8}{5} \times 7 \text{ units of} \\
\text{quantity } A & & \text{quantity } B
\end{array}
$$

That is,

$$
\begin{array}{ccc}
8 \text{ units of} & \longleftrightarrow & 11.2 \text{ units of} \\
\text{quantity } A & & \text{quantity } B
\end{array}
$$

Comment: The double number line suggests that zero on the A number line matches zero on the B number line. Is this correct? Is $x = 0$, $y = 0$ a data point that belongs to the table?

Yes! From

$$
\begin{array}{ccc}
5 \text{ units of} & \longleftrightarrow & 7 \text{ units of} \\
\text{quantity } A & & \text{quantity } B
\end{array}
$$

scale by a factor of zero to get

$$
\begin{array}{ccc}
0 \text{ units of} & \longleftrightarrow & 0 \text{ units of} \\
\text{quantity } A & & \text{quantity } B
\end{array}
$$

In general (0,0) is a valid data point for any two quantities in a proportional relationship.

Tables and equations

It is natural to represent the measures of quantities by letters of the alphabet that relate to the quantity being measured: A for the number of apples, d for the number of dollars, h for the height of the stack, and so forth. But in mathematics there is a strong predilection to repeatedly use the letters x and y for two unknown values. We now venture there as well.

Here again is the data of the previous section, with the two variables of concern renamed. We will presume that the two quantities represented here are indeed in a proportional relationship.

x	y
5	7
10	14
30	42
3	4.2
6	8.4

From the line

$$\text{10 units of the first quantity} \longleftrightarrow \text{14 units of the second quantity}$$

(or from any other line, for that matter) we see

$$\text{1 unit of the first quantity} \longleftrightarrow \text{1.4 units of the second quantity}$$

and so

$$\text{x units of the first quantity} \longleftrightarrow \text{1.4x units of the second quantity}$$

x	y
5	7
10	14
30	42
3	4.2
6	8.4
1	1.4
x	1.4x

Thus, we see that the variables satisfy the equation

$$y = 1.4x$$

That is, for any pair of values x and y that belong to a row of this table, the second value is sure to be 1.4 times as large as the first.

This argument shows that for any two quantities in a proportional relationship with measures represented by the variables x and y, specific instances of these measures are sure to satisfy

$$y = rx$$

for some fixed number r.

If

$$\text{a units of the first quantity} \longleftrightarrow \text{b units of the second quantity}$$

then

$$\text{1 units of the first quantity} \longleftrightarrow \text{$\frac{b}{a}$ units of the second quantity}$$

and so

$$\text{x units of the first quantity} \longleftrightarrow \text{$\left(\frac{b}{a}\right)x$ units of the second quantity}$$

of the second quantity.

x	y
a	b
1	b / a
x	(b / a) x

What do we mean by an equation?

People say that mathematics is a language. This is true. Since this article is being written in English, the language of mathematics is … English! (And if we were writing in Hindi or in Korean, the language of mathematics would be Hindi or Korean.)

Every mathematical statement is a sentence. For example, the statement

$$2 + 3 = 5$$

has a noun (the quantity "2 + 3"), a verb ("equals"), and an object (the quantity "5").

The statement

$$3 > 4 + 6$$

is also a sentence.

The first sentence happens to be a true sentence about numbers and the second a false sentence about numbers. As mathematics tends to focus on truth, it is interested in sentences that represent true statements about numbers.

The sentence

$$y = 0.6x$$

about two unspecified numbers called x and y, for instance, is neither true nor false as it stands: it all depends what specific two values one might assign to the two variables. If $x = 10$ and $y = 6$, then we will have a true sentence about numbers. If $x = 2$ and $y = 50$, then we will have a false sentence about numbers.

Given an equation, it is natural then to seek from it all the values of the variables that make the sentence a true sentence about numbers.

One collects from an equation all the data values that make the equation a true sentence about numbers.

For example, from $y = 0.6x$, we can create, by trial and error, a table of data values that each yield a true number sentence.

x	y
10	6
5	3
2	1.2

Question: If one triples the entries of the final row shown, do we generate a new pair of data values that also make the equation $y = 0.6x$ a true number sentence?

Yes! One checks that $x = 6$ and $y = 3.6$ do indeed fit the equation.

Question: In general, does all the data in this example scale in tandem?

We have that $x = 2$ and $y = 1.2$ make $y = 0.6x$ a true number sentence.

$$1.2 = 0.6 \times 2$$

If we scale each value by a factor k, are $x = 2k$ and $y = 1.2k$ sure to lead to a true number sentence as well?

In algebra we believe that multiplying both sides of an equation by a common factor does not alter the truth of that equation. So we see

$$(1.2)k = (2 \times 0.6)k$$

is a true number sentence. Since this can be rewritten

$$(1.2k) = 0.6 \times (2k)$$

we see indeed that $x = 2k$ and $y = 1.2k$ fit the equation $y = 0.6x$, too.

Data generated from an equation of the form $y = rx$ is sure to scale in tandem.

This is profound: we now have a complete and robust algebraic formulation of a proportional relationship between two covarying measurable quantities in a scenario.

The data arising from any proportional relationship satisfies an equation of the form $y = rx$ for some fixed constant r. And, conversely, the data that arises from any given equation of the form $y = rx$ is in a proportional relationship.

Examples

From

$$€100 \longleftrightarrow \$114$$

we obtain

$$€1 \longleftrightarrow \$1.14$$

and

$$€x \longleftrightarrow \$1.14x$$

We imagine scaling the number of euros by x, thus we need to scale (in tandem) the number of dollars by x. Thus, x represents the scale factor. And we see that the "exchange rate" from euro (x) to USD (y) is 1.14 and can use the formula $y = 1.14x$ to readily convert from euro to USD.

Alternatively, we could focus on USD and write

$$€\frac{100}{114} \approx 0.877 \longleftrightarrow \$1$$

and

$$€0.877x \longleftrightarrow \$x$$

and see that the USD to euro exchange rate is 0.877 with corresponding formula $y = 0.877x$ to convert from USD (x) to euro (y).

From

$$\$1\,\text{earned} \longleftrightarrow \$0.35\,\text{for the IRS}$$

we see that I pay taxes at a rate of 0.35, and we have the formula $y = 0.35x$ to compute the count of dollars to go to taxes (y) if I earn x dollars.

For the rising road to Wagga Wagga we have

$$1\,\text{mile} \longleftrightarrow 100\,\text{feet}$$

and we say that the road is rising at a rate of 100 feet per mile traversed. We have the general formula $y = 100x$ for computing rise (y feet) over x miles traversed.

For a train moving at constant speed

$$1\,\text{hour} \longleftrightarrow 65\,\text{feet}$$

we say that it is moving at a constant rate of 65 miles per hour. We have the formula $y = 65x$ for computing the miles (y) traversed for the number of hours of motion (x).

Definition: For two measurable quantities x and y in a proportional relationship, we can find a number r so that

$$\begin{array}{ccc} 1\,\text{unit of} & \longleftrightarrow & r\,\text{units of} \\ \text{quantity}\,x & & \text{quantity}\,y \end{array}$$

This number is called the *rate* (or in some curricula, the "unit rate") of the proportional relationship. We have that the data of the relationship satisfies the equation $y = rx$.

As the euro/USD exchange rate example shows, one can make choices as to which quantity to focus on as the "driving force" of the scenario: focusing on USD gives an exchange rate from USD to euro; focusing on euro gives an exchange rate from euro to USD.

In any scenario, one relies on context to decide which quantity in the scenario to focus on.

Comment: One might be able to recognize a "*y = rx*" relationship by reading across the rows of a table of data. For instance, for this data

x	y
6	9
10	15
2	3
5	7.5

we notice that each value in the right column is one-and-a-half times as large as its matching value in the left column, and so we see that this small data set is following an equation of the form *y* = 1.5*x*. If we have reason to believe this pattern persists for all possible data values, then we can say that the data is conforming to a proportional relationship.

The value *r* in a relationship identified algebraically via *y = rx* is often called the *constant of proportionality*.

The constant of proportionality is the rate.

Comment: One might instead notice that each pair of values in a row of the table produce equivalent ratio values when dividing one by the other. For instance, in the table we see that $\frac{9}{6}$ and $\frac{15}{10}$ and $\frac{3}{2}$ and $\frac{7.5}{5}$ each have value 1.5. This suggests that the equation
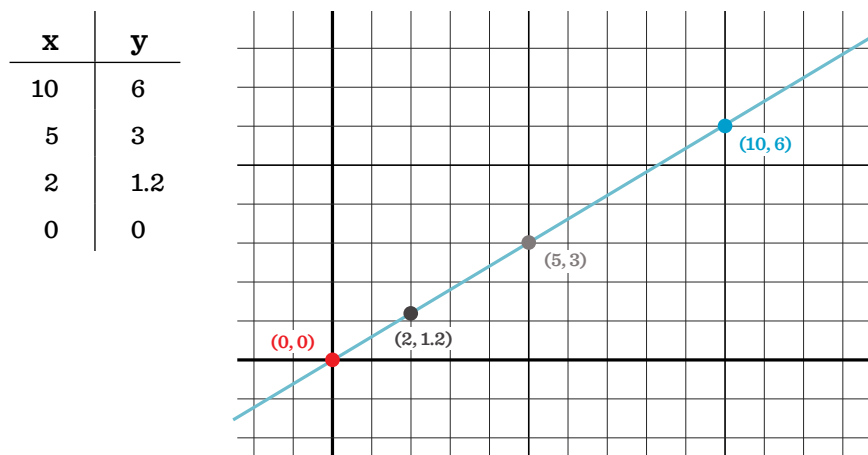
$$y/x = 1.5$$

is at play, which, of course, is equivalent to the equation *y* = 1.5*x*. (Unless the data point *x* = 0, *y* = 0 is listed in the table!)
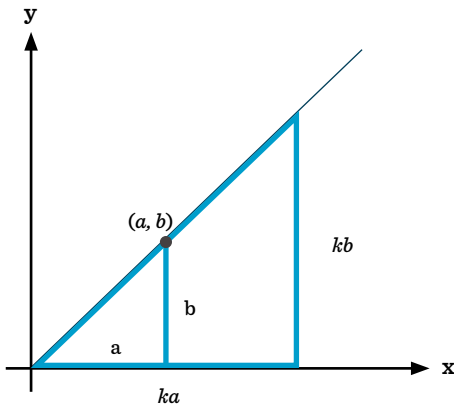
Graphs

An equation of the form $y = rx$ graphs as a straight line through the origin. And as any set of data arising from a proportional relationship follows an equation of this form, that data must plot as a straight line through the origin.

x	y
10	6
5	3
2	1.2
0	0

(10, 6)

(5, 3)

(0, 0)   (2, 1.2)

Conversely, any data that plots along a straight line through the origin follows an equation of the form $y = rx$ and so represents a proportional relationship.

One can also see this via the tools of geometry rather than the tools of algebra:

If (a, b) is a point on a line through the origin, then similar triangles show that (ka, kb) is also on that line.

So if

$$\begin{array}{ccc} a \text{ units of} & \longleftrightarrow & b \text{ units of} \\ \text{quantity } x & & \text{quantity } y \end{array}$$

then we have established that

$$\begin{array}{ccc} ka \text{ units of} & \longleftrightarrow & kb \text{ units of} \\ \text{quantity } x & & \text{quantity } y \end{array}$$

The data that arises from points along a line through the origin are in a proportional relationship.

The slope of the line though the origin arising from plotting data from a proportional relationship, $y = rx$, is r, the rate associated with the proportional relationship. The point $(1, r)$ lies on the line plot.

Comment: We have now seen the "rate" of a proportional relationship appear in multiple contexts.

i) When the measure of one quantity is scaled to unit value, the measure of the other quantity is r.

ii) r is the constant of proportionality of the algebraic equation that arises: $y = rx$.

iii) In each row of a data table, each value on the right is r times its matching value on the left.

iv) r is the slope of the line through the origin on which the plotted data sits.

v) The point $(1, r)$ appears in the plot of data values.

It is worth reflecting again for a moment to take stock of the various manifestations of this one number.

Direct variation

Science strives to supply structure and predictability to observed phenomena. This is most often accomplished by identifying the varying (and hopefully measurable) quantities in the scenario and writing an equation that holds true for possible values of the measures of those quantities. For example, Newton's law of gravitation states that the gravitational force F experienced between two objects of masses M_1 and M_2 at a distance of r units apart (following standard scientific units) always have values that make

$$F = G\,\frac{M_1 M_2}{r^2}$$

a true number sentence, where G here is a known fixed constant.

To understand or to test predicted equations, scientists might run tests or experiments, keeping all but two of the mentioned quantities fixed in measure and varying the value of one to observe the matching value of the other.

For example, we could (theoretically) test one aspect of Newton's law of gravitation by measuring the gravitational force on objects of different masses held a fixed distance from the sun. (That is, with M_1 and r set to be fixed in value throughout our tests.) The law says we should have

$$F = \left(\frac{GM_1}{r^2}\right) M_2$$

that is, that F and M_2 should be in a proportional relationship (with the fixed value of GM_1/r^2 the constant of proportionality).

Scientists will use the swift, asymmetrical notation

$$F \propto M_1$$

and say that they expect F to *vary directly* with M_1 to mean that these two quantities should scale in tandem (when all other quantities are held fixed). The asymmetry of the notation suggests that M_1 is the variable of focus, the one whose value they will vary in tests, to record the matching values of F that result. We also see in Newton's law of gravitation that $F \propto M_2$ and even that $F \propto (1/r^2)$ regarding the "reciprocal of the square of the distance" as a quantity to be measured in its own right!

This asymmetrical thinking often comes up in scenarios of a proportional relationship between two measurable quantities if it is natural to think of the measure of one quantity as the "driving force" of the scenario, with the measure of the other as dependent on these first measures. For example, in walking directly east at a constant speed of 5 miles per hour, the total distance I walk depends on the number of hours I've been walking. The distances one could walk are in a proportional relationship with the numbers of hours one can spend walking. But it feels natural to think of my time spent walking as the key factor here. To reflect this, people might say:

> Distance walked is *proportional to* time spent walking

Following other examples in this essay, it might also feel natural to say:

> The amount of money spent on Kewpie dolls is proportional to the number of dolls purchased.

> The amount of tax one pays is proportional to the total amount of money one earns.

> The amount one rises along that road to Wagga Wagga is proportional to the distance one drives along it.

In short, the phrase "is proportional to" indicates that the two measurable quantities mentioned are in a proportional relationship and, moreover, that the second quantity mentioned is considered the "driving force" in that scenario.

Inverse variation

If it takes ten volunteers four hours to pack and address 600 envelopes, then doubling the number of volunteers on the task should halve the time needed to complete it. Tripling the count should require one-third of the time. Reducing the count of volunteers to just one (a factor of ten) will likely require ten times the hours.

This scenario illustrates two measurable quantities—a count of volunteers and the time needed to complete the task—that seem to scale in "anti-tandem"!

Two measurable quantities in a scenario are said to be in an *inverse proportional relationship* if changing the measure of one quantity by a factor k causes the measure of the other to change by a factor $1/k$.

Example

In sharing a cake, the amount of cake each person receives is inversely proportional to the number of people sharing the cake: triple the number of people and each person receives a piece reduced to a third of the size.

A worked example

It takes 3 people 8 hours to wash all the windows of an office building. How many hours would it take 5 people to complete the task?

We have

$$3 \text{ people} \longleftrightarrow 8 \text{ hours}$$

Common sense tells us that that one person will take triple the time

$$1 \text{ person} \longleftrightarrow 24 \text{ hours}$$

and then that five people will take one-fifth of this time

$$5 \text{ people} \longleftrightarrow \frac{24}{5} = 4.8 \text{ hours}$$

The answer is 4.8 hours (four hours and 48 minutes).

If we denote the count of people on the job by x and the number of hours to complete the work by y, then from

$$1 \text{ person} \longleftrightarrow 24 \text{ hours}$$

we deduce

$$x \text{ people} \longleftrightarrow \frac{24}{x} \text{ hours}$$

x	y
3	8
1	24
5	4.8
x	$\frac{24}{x}$

We see we have the equation $y = \dfrac{24}{x}$.

We can extend this work and deduce

The data arising from any inverse proportional relationship satisfies an equation of the form $y = r / x$ for some fixed constant r. And, conversely, the data that arises from any given equation of the form $y = r / x$ is in an inverse proportional relationship.

Thus if quantities x and y are in an inverse proportional relationship, then $y \propto 1/k$. (Pause to take in what is being said here.)

Scientists say that *y varies inversely* with *x*.

Example
In Newton's law of gravitation we can say that the gravitational force two objects mutually experience varies directly with the mass of each object and varies inversely with the square of the distance between the two objects.

One can apply the thinking of this article to scenarios involving more than two measurable varying quantities. This next example is based on a popular teaser.

A mixed example
If 6 cats can catch 7 rats in 4 days, to the nearest half hour, how long does it take 1 cat to catch 1 rat?

We have

$$6 \text{ cats} \longleftrightarrow 7 \text{ rats} \longleftrightarrow 4 \text{ hours}$$

It seems we should assume that the count of cats varies inversely with time (if we double the count of cats, then the time needed to catch a fixed count of rats halves) and that the count of rats varies directly with time (doubling the count of rats to be caught by a fixed set of cats requires double the time).

Let's adjust the numbers to 1 cat and then to 1 rat, adjusting the time as we go along.

First, we observe that one-sixth of the count of cats will require six times the number of hours for a fixed count of rats.

$$1 \text{ cat} \longleftrightarrow 7 \text{ rats} \longleftrightarrow 24 \text{ hours}$$

Next we observe catching one-seventh of the count of rats will require only one-seventh of the number of hours.

$$1 \text{ cat} \longleftrightarrow 1 \text{ rat} \longleftrightarrow \frac{24}{7} \text{ hours}$$

And we are done! The answer is close to 3.5 hours.

Discussion: A real-life scenario
During a recent visit to Hong Kong, James noticed the exchange rate was

$$1 \text{ HKD} \longleftrightarrow 0.127 \text{ USD}$$

He rounded matters and held the exchange rate 100 HKD ≈ 13 USD in his head. (So he knew that spending 10 HKD was equivalent to spending close to $1.30, spending 200 HKD was close to spending $26, and so on.)

James discussed the exchange rate with a colleague who said she simply divided all the values by eight.

a. Explain why both methods, up to the rounding they each conducted, were equivalent.

b. Do you think it would be natural for James to say that the number of HKD is "proportional to" the number of USD, or the other way around? What might a Hong Kong resident say? Is the "proportional to" language even relevant here?

Summary

We hope that this article is a useful tool to help teachers and others get started on a journey to identify the meaningful ideas inherent in proportionality. Proportionality does not need to be cluttered, either in the curriculum or in the minds of learners. It is not a collection of disconnected procedures and ideas. In fact, as we attempted to show here, it makes perfect sense when we begin by focusing on ways of *thinking* (scaling in tandem), rather than ways of *doing*. You may have noticed that in all of this we did not use typical ways of doing, such as setting up a proportion or cross multiplying. It wasn't necessary, as simply thinking about the quantities did the work.

There is much more to learn and discuss, however. Some may want to think about connections to the development of multiplication and division. Others may want to examine how this thinking connects powerfully as a cornerstone to subsequent mathematics. A few may want to ponder existing definitions (and debates) about ratios and rates and how those may or may not be compatible with this presentation. We hope to have sparked *some* interest in further learning. More importantly, though, we hope that students can see the beauty and connectedness in the mathematics they experience in school. This is one place we can make that happen.

References

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